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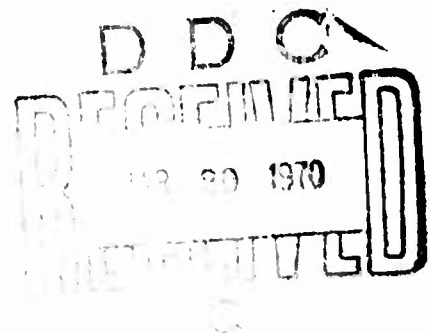
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REPORT NO. 1471

## TRANSIENT HEATING OF THIN PLATES

by

Harold J. Breaux  
Palmer R. Schlegel



February 1970

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**BALLISTIC RESEARCH LABORATORIES**

**REPORT NO. 1471**

**FEBRUARY 1970**

**TRANSIENT HEATING OF THIN PLATES**

**Harold J. Breaux  
Palmer R. Schlegel**

**Applied Mathematics Division**

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REPORT NO. 1471

HJBreaux/PRSchlegel  
Aberdeen Proving Ground, Md.  
February 1970

TRANSIENT HEATING OF THIN PLATES

ABSTRACT

Two classes of heating problems involving the heating of thin plates over a portion of their surface are considered. Such problems have numerous applications, one such application being the determination of the temperature rise in materials upon which a laser beam is directed. The heating effects caused by a laser beam fits into the class of problems for which the heating source has circular symmetry. The second class of problems arises when the source term has symmetry about a line on the surface. Solutions in the form of a definite integral are obtained for both classes of problems for a general time independent source of unspecified spatial distribution. Several solutions in closed form are obtained for particular source functions and for particular values of the various parameters of the problem. For the case of a disk source an error bound useful in the numerical quadrature of the integral solution is obtained.

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# LIST OF SYMBOLS

$c$	specific heat
$\text{erf}(x)$	error function
$\text{erfc}(x)$	complementary error function
$E_1(x)$	exponential integral function
$h$	heat transfer coefficient
$I_n(x)$	modified Bessel function of the first kind
$J_n(x)$	Bessel function
$K_n(x)$	modified Bessel function of the third kind
$k$	thermal conductivity
$q(r)$	dimensionless circular symmetric source function, See Eq. (9)
$q(x)$	dimensionless line symmetrical source function
$\bar{q}_c(\sigma)$	Fourier cosine transform of $q(x)$
$\bar{q}_H(\sigma)$	zero order Hankel transform of $q(r)$
$Q(R, \beta)$	source function
$Q_o$	flux intensity at center of heat source, $Q(o, \beta)$
$r$	dimensionless radial coordinate, $R/\beta$
$R$	radial coordinate
$t$	dimensionless time, See Eq. (5)
$T$	dimensionless temperature, See Eq. (7)
$\bar{u}_c(\sigma)$	Fourier cosine transform of $u(x)$ , See Eq. (70)
$\bar{v}_H(\sigma)$	zero order Hankel transform of $v(r)$ , See Eq. (16)
$x$	cartesian coordinate
$\alpha$	thermal diffusivity
$\beta$	parameter in source function
$\Gamma(x)$	gamma function
$\delta$	plate thickness
$\epsilon$	dimensionless heat transfer coefficient, See Eq. (8)
$\theta$	temperature
$\theta_o$	initial temperature, temperature of cooling medium

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LIST OF SYMBOLS (cont'd)

$\rho$	density
$\sigma$	integral transform variable
$\tau$	time

## I. INTRODUCTION

Several types of conduction heat transfer applications involve the determination of transient temperature in a thin plate heated over a portion of its surface, the surface being simultaneously cooled through surface convection. Typical applications include the heating of thin metal plates with a laser or incendiary, drilling holes in metal, acetylene burning, arc welding and friction heating of machine parts. Solutions for uniform disk and uniform strip sources were obtained by Thomas<sup>1\*</sup>, the solutions being in the form of definite integrals for the most general cases. For special cases such as no cooling, steady state, or at the center of the heated area, Thomas reduced the general integral solutions to the form of functions for which tables exist. In this report the results cited above are generalized to include solutions for all time independent source functions meeting certain fairly general criteria. Additionally several new results in closed form are obtained. For the case of a disk source the general integral solution is in the form of a definite integral on the interval  $(0, \infty)$ . For this case an error bound is obtained which yields an upper bound on the error incurred when this integral is approximated by numerical quadrature over finite limits.

## II. CIRCULAR SYMMETRIC SOURCES

We assume the thin plate of thickness  $\delta$  has constant thermal properties, the cooling effect can be represented by Newton's Law of Cooling and the temperature gradient across the thickness of the thin plate is negligible. We also assume that the initial temperature and the temperature of the cooling medium is constant at  $\theta = \theta_0$ . Under these conditions and for a circular symmetric source, Fig. 1, the applicable partial differential equation, initial condition and boundary conditions are the following:

$$\rho c \frac{\partial \theta}{\partial \tau} = \frac{k}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \theta}{\partial R} \right) - \frac{2h}{\delta} (\theta - \theta_0) + \frac{Q(R, \beta)}{\delta}, \quad R > 0, \tau > 0 \quad (1)$$

$$\theta = \theta_0 \quad R \geq 0, \tau = 0 \quad (2)$$

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\* References are listed on page 24.



# CIRCULAR SYMMETRIC SOURCE

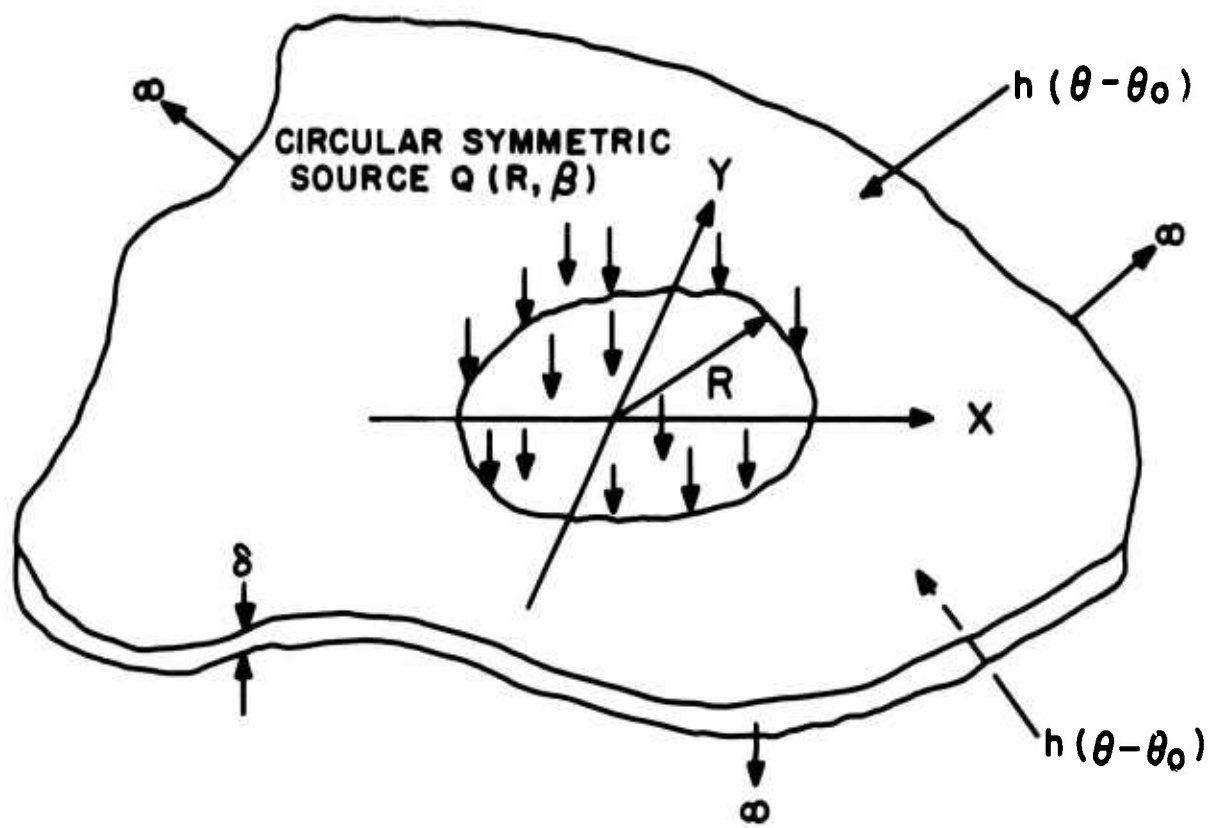


Fig. 1

$$\theta = \frac{\partial \theta}{\partial R} = 0 \quad R \rightarrow \infty, \tau \geq 0 \quad (3)$$

$$\theta < \infty \quad R = 0, \tau \geq 0. \quad (4)$$

In the usual fashion it is desirable to work with dimensionless variables. If we let

$$t = \alpha \tau \beta^{-2}, \quad (5)$$

$$r = R \beta^{-1}, \quad (6)$$

$$T = k \delta (\theta - \theta_0) \beta^{-2} Q_0^{-1}, \quad (7)$$

$$\epsilon^2 = 2 h \beta^2 k^{-1} \delta^{-1}, \quad (8)$$

$$q(r) = Q(\beta r, \beta) Q_0^{-1}, \quad (9)$$

where

$$Q_0 = Q(0, \beta), \quad (10)$$

$$\alpha = k \rho^{-1} c^{-1} \quad (11)$$

then the problem can be expressed in the following dimensionless form:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \epsilon^2 T + q(r), \quad r > 0, t > 0 \quad (12)$$

$$T = 0, \quad r \geq 0, t = 0 \quad (13)$$

$$T = \frac{\partial T}{\partial r} = 0, \quad r \rightarrow \infty, t \geq 0 \quad (14)$$

$$T < \infty \quad r = 0, t \geq 0. \quad (15)$$

The technique employed to obtain a solution to (12) - (15) is the integral transform method. The zero order Hankel transform of any arbitrary function  $v(r)$  is defined by

$$\bar{v}_H(\sigma) = \int_0^\infty r v(r) J_0(\sigma r) dr, \quad (16)$$

where its inverse transform is given by

$$v(r) = \int_0^\infty \sigma \bar{v}_H(\sigma) J_0(\sigma r) d\sigma. \quad (17)$$

By taking the zero order Hankel transform of (12), we obtain the subsidiary equation

$$\frac{d\bar{T}_H(\sigma, t)}{dt} = -(\sigma^2 + \epsilon^2)\bar{T}_H(\sigma, t) + \bar{q}_H(\sigma). \quad (18)$$

This result arises from the identity (see Carslaw and Jaeger<sup>2</sup>)

$$\int_0^\infty r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) r J_0(\sigma r) dr = -\sigma^2 \int_0^\infty r v J_0(\sigma r) dr, \quad (19)$$

which holds if  $v(0)$  is finite and

$$v(r) = \frac{\partial v(r)}{\partial r} \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (20)$$

From (13) and (16) we have

$$\bar{T}_H(\sigma, 0) = 0. \quad (21)$$

The solution of (18) subject to (21) is

$$\bar{T}_H(\sigma, t) = \bar{q}_H(\sigma) (\sigma^2 + \epsilon^2)^{-1} [1 - \exp[-t(\sigma^2 + \epsilon^2)]]]. \quad (22)$$

From (22) and (17) we have

$$T(r, t) = \int_0^\infty \bar{q}_H(\sigma) (\sigma^2 + \epsilon^2)^{-1} \sigma J_0(\sigma r) [1 - \exp[-t(\sigma^2 + \epsilon^2)]] d\sigma. \quad (23)$$

This solution is valid for all circular symmetric sources for which a Hankel transform  $\bar{q}_H(\sigma)$  is defined. For the most general cases the above integral cannot be reduced further and one must resort to numerical quadrature for obtaining numerical values.

#### A. Gaussian Circular Source

A Gaussian circular source is defined by

$$Q(R, \beta) = Q_0 e^{-R^2 \beta^{-2}}, \quad (24)$$

which in dimensionless form reduces to

$$q(r) = e^{-r^2}. \quad (25)$$

We consider the case for no cooling,  $\epsilon = 0$ . For this case  $\bar{q}_H(\sigma)$  is given by (10), p.29, Erdelyi<sup>\*</sup>, et. al<sup>3</sup>, Vol.2

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\* Hereafter the two volumes of this reference will be referred to as [EMOT1] and [EMOT2].

$$\bar{q}_H(\sigma) = \frac{1}{2}e^{-\frac{1}{4}\sigma^2}. \quad (26)$$

The general solution (23) reduces to

$$T(r,t) = \frac{1}{2} \int_0^\infty \sigma^{-1} J_0(\sigma r) \left[ e^{-\frac{1}{4}\sigma^2} - e^{-\sigma^2(t+\frac{1}{4})} \right] d\sigma. \quad (27)$$

To evaluate this integral we make use of the identity

$$\int_0^\infty \sigma^{-1} e^{-a\sigma^2} J_0(\sigma r) d\sigma = \frac{1}{2}(\pi/a)^{\frac{1}{2}} \int_r^\infty e^{-\sigma^2(8a)^{-1}} I_{\frac{1}{2}}[\sigma^2(8a)^{-1}] d\sigma, \quad (28)$$

which is derived by combining (7) p.5 and (8) p.29 of [EMOT2]. We also have the identity

$$I_{\frac{1}{2}}(z) = (2\pi z)^{-\frac{1}{2}}(e^z - e^{-z}). \quad (29)$$

Hence

$$\int_0^\infty \sigma^{-1} e^{-a\sigma^2} J_0(\sigma r) d\sigma = \int_r^\infty \sigma^{-1} (1 - e^{-\frac{1}{4}\sigma^2 a^{-1}}) d\sigma. \quad (30)$$

By letting  $y = \frac{1}{4}\sigma^2 a^{-1}$ , we obtain the following identity:

$$\begin{aligned} \int_r^\infty \sigma^{-1} e^{-\frac{1}{4}\sigma^2 a^{-1}} d\sigma &= \frac{1}{2} \int_{\frac{1}{4}r^2 a^{-1}}^\infty e^{-y} y^{-1} dy \\ &= \frac{1}{2} E_1\left(\frac{1}{4}r^2 a^{-1}\right). \end{aligned} \quad (31)$$

From (27), (30) and (31) we obtain the final result

$$T(r,t) = \frac{1}{4} \{ E_1[r^2(4t+1)^{-1}] - E_1(r^2) \}. \quad (32)$$

We note that this solution is unbounded for  $r = 0$ ; thus, we treat this case separately. For  $r = 0$  in (27), we obtain

$$\begin{aligned} T(0,t) &= \lim_{r \rightarrow 0} \frac{1}{4} \left[ \int_{r^2/(4t+1)}^\infty e^{-y} y^{-1} dy - \int_{r^2}^\infty e^{-y} y^{-1} dy \right] \\ &= \lim_{r \rightarrow 0} \frac{1}{4} \int_{r^2/(4t+1)}^{r^2} e^{-y} y^{-1} dy. \end{aligned} \quad (33)$$

Since the integral is to be evaluated near the origin, we can let

$$e^{-y} = 1 - y + O(y^2) \quad (34)$$

obtaining

$$\begin{aligned}
 T(0,t) &= \frac{1}{4} \lim_{r \rightarrow 0} \int_{r^2/(4t+1)}^{r^2} [(1-y)y^{-1} + 0(y)] dy \\
 &= \frac{1}{4} \ln(4t+1) + \lim_{r \rightarrow 0} \left[ 0(y^2) \Big|_{r^2/(4t+1)}^{r^2} \right] \\
 &= \frac{1}{4} \ln(4t+1).
 \end{aligned} \tag{35}$$

Hence, the solution for a Gaussian circular source and no cooling is given by

$$T(r,t) = \begin{cases} \frac{1}{4} \{ E_1[r^2(4t+1)^{-1}] - E_1(r^2) \}, & r > 0 \\ \frac{1}{4} \ln(4t+1), & r = 0. \end{cases} \tag{36}$$

#### B. Uniform Disk Source

A uniform disk source is defined by

$$Q(R,\beta) = \begin{cases} Q_0, & R \leq \beta \\ 0, & R > \beta, \end{cases} \tag{37}$$

which in dimensionless form reduces to

$$q(r) = \begin{cases} 1, & r \leq 1 \\ 0, & r > 1. \end{cases} \tag{38}$$

The zero order Hankel transform of (38) is given by (3), p.47 of [EMOT2].

$$\bar{q}_H(\sigma) = J_1(\sigma)/\sigma. \tag{39}$$

For this case the general solution reduces to

$$T(r,t) = \int_0^\infty J_0(\sigma r) J_1(\sigma) (\sigma^2 + \epsilon^2)^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}] d\sigma, \tag{40}$$

an integral solution obtained by Thomas<sup>1</sup>. In this form the above integral cannot be reduced further. For steady state the integral reduces to

$$T(r,\infty) = \int_0^\infty J_0(\sigma r) J_1(\sigma) (\sigma^2 + \epsilon^2)^{-1} d\sigma. \tag{41}$$

For  $r > 1$  this integral is given by (12), p.49 of [EMOT2], namely,

$$T(r, \infty) = \epsilon^{-1} I_1(\epsilon) K_0(\epsilon r), \quad r > 1. \quad (42)$$

For  $r < 1$  we delete the transient term in the original partial differential equation and obtain

$$r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) - \epsilon^2 T + 1 = 0. \quad (43)$$

The homogeneous equation is one of the standard Bessel types which has solutions  $I_0(\epsilon r)$  and  $K_0(\epsilon r)$ . We can eliminate  $K_0(\epsilon r)$ , since we expect a finite temperature at  $r = 0$ . The nonhomogeneous equation thus has a solution

$$T(r, \infty) = \epsilon^{-2} + C I_0(\epsilon r), \quad r < 1. \quad (44)$$

We determine the constant  $C$  by forcing the solution to be continuous at  $r = 1$ , that is, equating (42) and (44) at  $r = 1$ . We obtain

$$C = \epsilon^{-1} [I_1(\epsilon) K_0(\epsilon) - \epsilon^{-1}] [I_0(\epsilon)]^{-1}. \quad (45)$$

By employing the Wronskian

$$I_1(\epsilon) K_0(\epsilon) + I_0(\epsilon) K_1(\epsilon) = \epsilon^{-1}, \quad (46)$$

we have

$$C = -\epsilon^{-1} K_1(\epsilon). \quad (47)$$

This yields

$$T(r, \infty) = \epsilon^{-2} - \epsilon^{-1} K_1(\epsilon) I_0(\epsilon r) \quad (48)$$

The steady state solution for a disk source with cooling is thus given by

$$T(r, \infty) = \begin{cases} \epsilon^{-2} [1 - \epsilon K_1(\epsilon) I_0(\epsilon r)], & r \leq 1 \\ \epsilon^{-1} I_1(\epsilon) K_0(\epsilon r), & r > 1. \end{cases} \quad (49)$$

For the general case (40) can be approximated by an integral over finite limits. Let

$$T(r, t) = \int_0^U J_0(\sigma r) J_1(\sigma) (\sigma^2 + \epsilon^2)^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}] d\sigma + E. \quad (50)$$

We will now determine a  $U$  such that  $\mathcal{E}$  is bounded by some a priori acceptable error. First we determine a bound for  $J_0(z)$  and  $J_1(z)$ \*. From Watson<sup>4</sup>, p.74

$$J_n(z) = \frac{1}{2} [H_n^{(1)}(z) + H_n^{(2)}(z)], \quad (51)$$

and Watson p. 168

$$H_n^{(1)}(z) = (2/\pi z)^{\frac{1}{2}} e^{i\theta} / \Gamma(n + \frac{1}{2}) \int_0^\infty e^{-u} u^{n-\frac{1}{2}} (1 + \frac{1}{2}iu/z)^{n-\frac{1}{2}} du,$$

$$H_n^{(2)}(z) = (2/\pi z)^{\frac{1}{2}} e^{-i\theta} / \Gamma(n + \frac{1}{2}) \int_0^\infty e^{-u} u^{n-\frac{1}{2}} (1 - \frac{1}{2}iu/z)^{n-\frac{1}{2}} du, \quad (52)$$

where  $\theta = z - n\pi/2 - \pi/4$ , and  $2n$  is not an odd positive integer. Since

$$|1 - iu/2z| = |1 + iu/2z| = [1 + (u/2z)^2]^{\frac{1}{2}}, \quad (53)$$

then

$$|J_n(z)| \leq (2/\pi z)^{\frac{1}{2}} [\Gamma(n + \frac{1}{2})]^{-1} \int_0^\infty e^{-u} [u [1 + (u/2z)^2]^{\frac{1}{2}}]^{n-\frac{1}{2}} du. \quad (54)$$

For  $n = 0$ , we have

$$|J_0(z)| \leq (2/\pi z)^{\frac{1}{2}} [\Gamma(\frac{1}{2})]^{-1} \int_0^\infty u^{-\frac{1}{2}} e^{-u} [1 + (u/2z)^2]^{-\frac{1}{4}} du$$

$$\leq (2/\pi z)^{\frac{1}{2}} [\Gamma(\frac{1}{2})]^{-1} \int_0^\infty u^{\frac{1}{2}-1} e^{-u} du = (2/\pi z)^{\frac{1}{2}}. \quad (55)$$

For  $n = 1$ , where we assume  $z \geq \frac{1}{2}$ , we obtain

$$|J_1(z)| \leq (2/\pi z)^{\frac{1}{2}} [\Gamma(\frac{3}{2})]^{-1} \int_0^\infty u^{\frac{1}{2}} [1 + (u/2z)^2]^{\frac{1}{4}} e^{-u} du$$

$$\leq (2/\pi z)^{\frac{1}{2}} [\Gamma(\frac{3}{2})]^{-1} \int_0^\infty [u^2 + u^4]^{\frac{1}{4}} e^{-u} du$$

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\* The bounds derived here are generalized and the derivation is put forth in more detail in a forthcoming report by the authors.

$$\begin{aligned}
&\leq (2/\pi z)^{\frac{1}{2}} [\Gamma(\frac{3}{2})]^{-1} \left[ \int_0^1 2^{\frac{1}{4}} e^{-u} du + \int_1^{\infty} 2^{\frac{1}{4}} u e^{-u} du \right] \\
&\leq 2^{\frac{7}{4}} (1 + e^{-1}) \pi^{-1} (z)^{-\frac{1}{2}}.
\end{aligned} \tag{56}$$

Now in (50)

$$|E| \leq \int_U^{\infty} |J_0(\sigma r)| |J_1(\sigma)| (\sigma^2 + \epsilon^2)^{-1} d\sigma. \tag{57}$$

If we apply the bounds (55) and (56) to (57), this yields (assuming  $r > 0$ )

$$|E| \leq A B r^{-\frac{1}{2}} \int_U^{\infty} [\sigma(\sigma^2 + \epsilon^2)]^{-1} d\sigma, \tag{58}$$

where

$$A = 2^{\frac{7}{4}} (1 + e^{-1}) \pi^{-1} \tag{59}$$

and

$$B = (2/\pi)^{\frac{1}{2}}. \tag{60}$$

This implies that

$$|E| \leq \frac{1}{2} A B r^{-\frac{1}{2}} \epsilon^{-2} \ln[(U^2 + \epsilon^2) U^{-2}]. \tag{61}$$

If we solve for  $U$ , we obtain the limit of integration, that is,

$$U = \epsilon [\exp\{2 r^{\frac{1}{2}} \epsilon^2 |E| A^{-1} B^{-1}\} - 1]^{-\frac{1}{2}}, \tag{62}$$

where  $E$  is the error. For the case when  $r = 0$ , we use the fact  $J_0(0) = 1$ . Hence

$$\begin{aligned}
|E| &\leq A \int_U^{\infty} \sigma^{-\frac{1}{2}} \sigma^{-2} d\sigma \\
&\leq \frac{2}{3} A U^{-\frac{3}{2}},
\end{aligned} \tag{63}$$

or

$$U = (\frac{2}{3} A |E|^{-1})^{\frac{2}{3}}. \tag{64}$$

### III. SOURCES SYMMETRICAL ABOUT A LINE

The partial differential equation, initial and boundary conditions governing this case in dimensionless form are

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - \epsilon^2 T + q(x), \quad x > 0, \quad t > 0 \tag{65}$$



$$T = 0, \quad x \geq 0, \quad t = 0 \quad (66)$$

$$\frac{\partial T}{\partial x} = 0, \quad x = 0, \quad t > 0 \quad (67)$$

$$T = \frac{\partial T}{\partial x} = 0, \quad x \rightarrow \infty, \quad t \geq 0. \quad (68)$$

Condition (67) arises from the assumption that the heat source is symmetrical about the line  $x = 0$ , that is,

$$q(x) = q(-x). \quad (69)$$

Again we employ the integral transform method for obtaining a solution. We define the Fourier cosine transform of an arbitrary function  $u(x)$  as

$$\bar{u}_c(\sigma) \equiv (2/\pi)^{\frac{1}{2}} \int_0^\infty u(x) \cos(\sigma x) dx, \quad (70)$$

and its inverse transform by

$$u(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \bar{u}_c(\sigma) \cos(\sigma x) d\sigma. \quad (71)$$

Employing the identity

$$(2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos(\sigma x) dx = -\sigma^2 \bar{u}_c(\sigma) - (2/\pi)^{\frac{1}{2}} \frac{\partial u}{\partial x} \Big|_{x=0} \quad (72)$$

and taking the Fourier cosine transform of (65), we obtain the subsidiary equation

$$\frac{d\bar{T}_c(\sigma, t)}{dt} = -(\sigma^2 + \epsilon^2) \bar{T}_c(\sigma, t) + \bar{q}_c(\sigma), \quad (73)$$

which is subject to

$$\bar{T}_c(\sigma, 0) = 0. \quad (74)$$

As before the solution is

$$\bar{T}_c(\sigma, t) = \bar{q}_c(\sigma) [\sigma^2 + \epsilon^2]^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}]. \quad (75)$$

Taking the inverse Fourier cosine transform, we obtain

$$T(r, t) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \bar{q}_c(\sigma) [\sigma^2 + \epsilon^2]^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}] \cos(\sigma x) d\sigma. \quad (76)$$

This solution is valid for all sources symmetrical about the line  $x = 0$ , for which a Fourier cosine transform exists and for which (69) holds.

### A. Uniform Strip Source

A uniform strip source is defined by

$$Q(x, \beta) = \begin{cases} Q_0, & x \leq \beta \\ 0, & x > \beta. \end{cases} \quad (77)$$

In dimensionless form this equation becomes

$$q(x) = \begin{cases} 1, & x \leq 1 \\ 0, & x > 1. \end{cases} \quad (78)$$

The Fourier cosine transform of (78) is

$$\bar{q}_c(\sigma) = (2/\pi)^{\frac{1}{2}} \sigma^{-1} \sin(\sigma). \quad (79)$$

Inserting (79) into (76), we obtain the integral solution obtained by Thomas<sup>1</sup>

$$T(x, t) = (2/\pi) \int_0^\infty \sigma^{-1} \sin(\sigma) [\sigma^2 + \epsilon^2]^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}] \times \\ \times \cos(x\sigma) d\sigma. \quad (80)$$

Let

$$T(x, t) = T_1(x) - T_2(x, t), \quad (81)$$

where

$$T_1(x) = (2/\pi) \int_0^\infty \sigma^{-1} \sin(\sigma) [\sigma^2 + \epsilon^2]^{-1} \cos(x\sigma) d\sigma \quad (82)$$

and

$$T_2(x, t) = (2/\pi) e^{-\epsilon^2 t} \int_0^\infty \sigma^{-1} \sin(\sigma) [\sigma^2 + \epsilon^2]^{-1} e^{-t\sigma^2} \cos(x\sigma) d\sigma. \quad (83)$$

From (4), p.19, of [EMOT1]

$$T_1(x) = \begin{cases} \epsilon^{-2} [1 - e^{-\epsilon} \cosh(\epsilon x)], & 0 \leq x \leq 1 \\ \epsilon^{-2} e^{-\epsilon x} \sinh(\epsilon), & 1 < x. \end{cases} \quad (84)$$

In evaluating (83), we first consider the related integral

$$w(x, t) = (2/\pi) e^{-\epsilon^2 t} \int_0^\infty \sigma^{-1} [\sigma^2 + \epsilon^2]^{-1} e^{-t\sigma^2} \sin(x\sigma) d\sigma \quad (85)$$

$$= (2/\pi) \epsilon^{-2} e^{-t\epsilon^2} \int_0^\infty [\sigma^{-1} - \sigma(\sigma^2 + \epsilon^2)^{-1}] e^{-t\sigma^2} \sin(x\sigma) d\sigma. \quad (86)$$

By employing (21), p.73 and (26), p.74 of [EMOT1], we obtain

$$w(x,t) = \epsilon^{-2} e^{-t\epsilon^2} \operatorname{erf}\left(\frac{1}{2}xt^{-\frac{1}{2}}\right) + \frac{1}{2}\epsilon^{-2} \left[ e^{\epsilon x} \operatorname{erfc}\left(\epsilon t^{\frac{1}{2}} + \frac{1}{2}xt^{-\frac{1}{2}}\right) - e^{-\epsilon x} \operatorname{erfc}\left(\epsilon t^{\frac{1}{2}} - \frac{1}{2}xt^{-\frac{1}{2}}\right) \right]. \quad (87)$$

It can be shown that

$$T_2(x,t) = \frac{1}{2}[w(1+x,t) + w(1-x,t)]. \quad (88)$$

If we collect these results, the solution for a uniform strip source is given by

$$T(x,t) = T_1(x) - T_2(x,t), \quad (89)$$

where

$$T_1(x) = \begin{cases} \epsilon^{-2} [1 - e^{-\epsilon \cosh(\epsilon x)}], & 0 \leq x \leq 1 \\ \epsilon^{-2} e^{-\epsilon x} \sinh(\epsilon), & 1 < x \end{cases} \quad (90)$$

and

$$T_2(x,t) = \frac{1}{2}\epsilon^{-2} \left\{ e^{-t\epsilon^2} \left[ \operatorname{erf}\left\{\frac{1}{2}(1+x)t^{-\frac{1}{2}}\right\} - \operatorname{erf}\left\{\frac{1}{2}(1-x)t^{-\frac{1}{2}}\right\} \right] + \frac{1}{2} \left[ e^{\epsilon(1+x)} \operatorname{erfc}\left\{\epsilon t^{\frac{1}{2}} + \frac{1}{2}(1+x)t^{-\frac{1}{2}}\right\} - e^{-\epsilon(1+x)} \operatorname{erfc}\left\{\epsilon t^{\frac{1}{2}} - \frac{1}{2}(1+x)t^{-\frac{1}{2}}\right\} + e^{\epsilon(1-x)} \operatorname{erfc}\left\{\epsilon t^{\frac{1}{2}} + \frac{1}{2}(1-x)t^{-\frac{1}{2}}\right\} - e^{-\epsilon(1-x)} \operatorname{erfc}\left\{\epsilon t^{\frac{1}{2}} - \frac{1}{2}(1-x)t^{-\frac{1}{2}}\right\} \right] \right\}. \quad (91)$$

## B. GAUSSIAN STRIP SOURCE

We define a "Gaussian strip source" by

$$Q(x,\beta) = Q_0 e^{-x^2/\beta^2}. \quad (92)$$

In dimensionless form this source is given by

$$q(x) = e^{-x^2} \quad (93)$$

and its Fourier cosine transform, which is obtained from (11), p. 15 of [EMOT1] is

$$\bar{q}_c(\sigma) = 2^{-\frac{1}{2}} e^{-\sigma^2/4}. \quad (94)$$

The solution for a strip source is thus

$$T(x,t) = \pi^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{1}{4}\sigma^2} [\sigma^2 + \epsilon^2]^{-1} [1 - \exp\{-t(\sigma^2 + \epsilon^2)\}] \cos(\sigma x) d\sigma. \quad (95)$$

Employing (15), p. 15 of [EMOT1] we obtain

$$T(x,t) = \frac{1}{4}\epsilon^{-1} \pi^{\frac{1}{2}} e^{\frac{1}{4}\epsilon^2} \left\{ e^{-\epsilon x} [\operatorname{erfc}(\frac{1}{2}\epsilon - x) - \operatorname{erfc}\{\frac{1}{2}\epsilon(4t+1)^{\frac{1}{2}} - x(4t+1)^{-\frac{1}{2}}\}] + e^{\epsilon x} [\operatorname{erfc}(\frac{1}{2}\epsilon + x) - \operatorname{erfc}\{\frac{1}{2}\epsilon(4t+1)^{\frac{1}{2}} + x(4t+1)^{-\frac{1}{2}}\}] \right\}. \quad (96)$$

Note that in the solutions for sources symmetrical about a line, the domain of interest,  $-\infty < x < \infty$ , has been restricted to  $0 \leq x < \infty$  for mathematical simplicity. This causes no loss of generality, since the solutions obtained can be extended to negative  $x$  simply by employing the symmetry condition  $T(-x) = T(x)$ ,  $x \geq 0$ .

#### IV. GRAPHICAL RESULTS

Figure 2 is a graphical presentation of the dimensionless temperature rise at the center of the heated uniform strip divided by the dimensionless time as a function of the dimensionless group  $\beta(\alpha\tau)^{-\frac{1}{2}}$ . By use of the dimensionless variables defined in the List of Symbols the effect on the temperature rise above ambient,  $(\theta - \theta_0)$ , can be determined as a function of any of the variables  $k$ ,  $\delta$ ,  $h$ ,  $Q_0$ ,  $\alpha$ ,  $\tau$ .

Figure 3 is a graph of the same quantities as Figure 2 for no surface cooling and is a comparison of the transient temperature rise above ambient at the source center for the sources listed.

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The authors' interest in the class of problems discussed in this report arose from a problem presented by Mr. Frank J. Allen of the Terminal Ballistics Laboratory related to the heating of "thin" materials by laser beams.

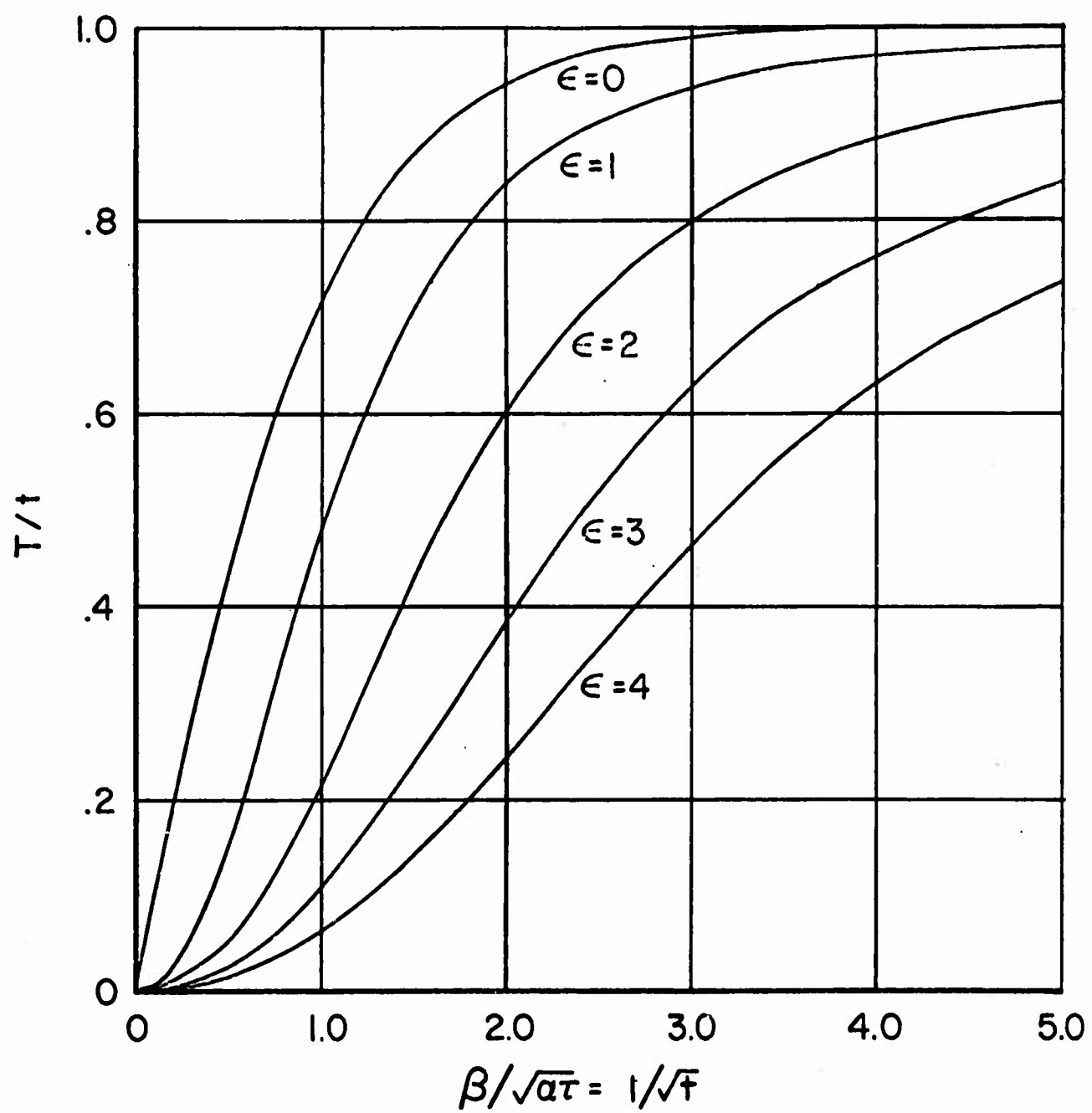


Fig. 2. Effect of strip width, heating time, and heat transfer coefficient on maximum temperature of thin plate heated by a uniform strip source.

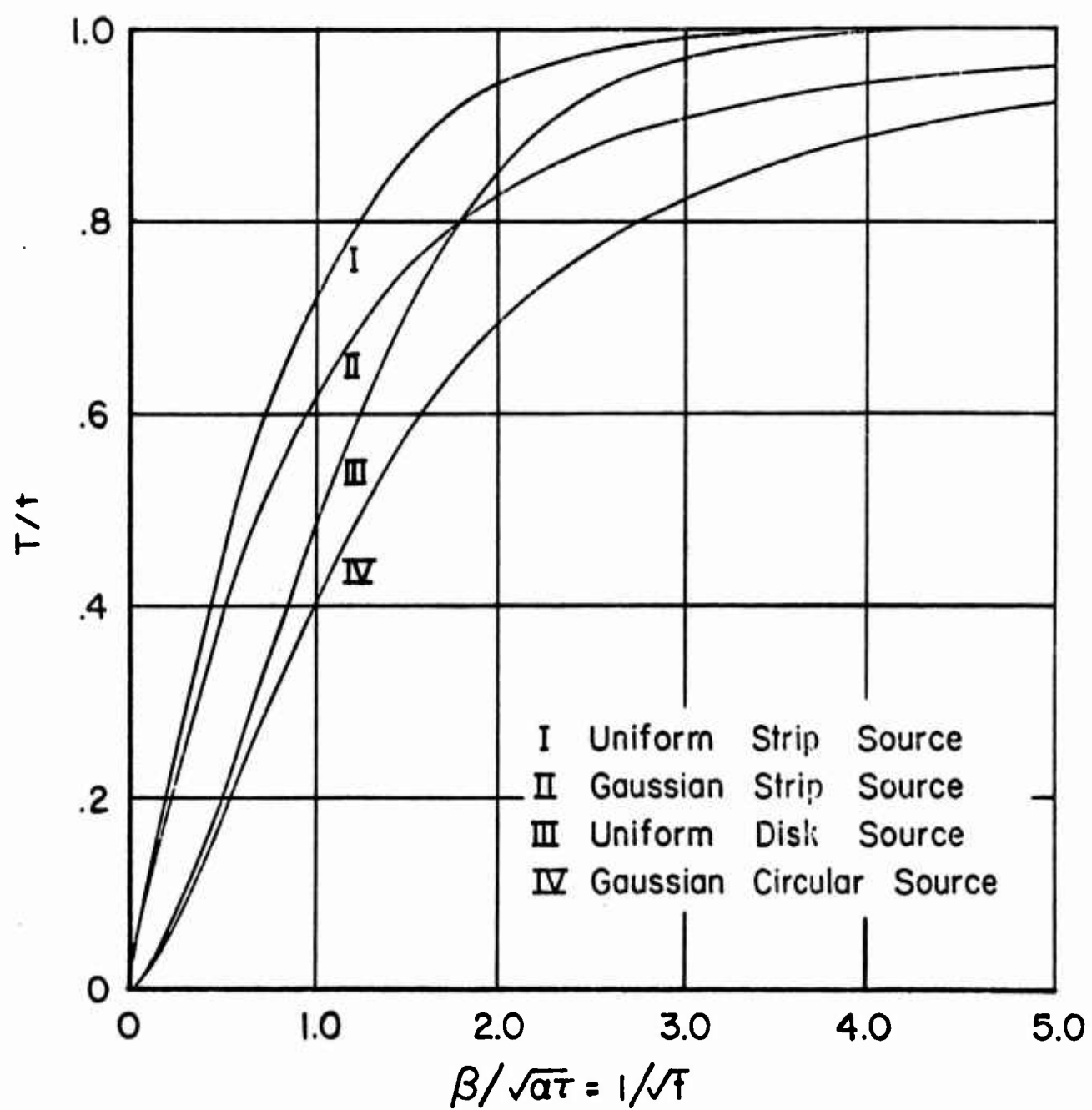


Fig. 3. Effect of source size, source type, and heating time on maximum temperature of thin plate with no cooling ( $\epsilon=0$ ).

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13. ABSTRACT Two classes of heating problems involving the heating of thin plates over a portion of their surface are considered. Such problems have numerous applications, one such application being the determination of the temperature rise in materials upon which a laser beam is directed. The heating effects caused by a laser beam fits into the class of problems for which the heating source has circular symmetry. The second class of problems arises when the source term has symmetry about a line on the surface. Solutions in the form of a definite integral are obtained for both classes of problems for a general time independent source of unspecified spatial distribution. Several solutions in closed form are obtained for particular source functions and for particular values of the various parameters of the problem. For the case of a disk source an error bound useful in the numerical quadrature of the integral solution is obtained.		

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